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FINDING INVARIANT TORI WITH POINCARE'S MAP

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Abstract. We consider the existence problem of invariant tori for quasi-periodic equation. We regard quasi-periodic functions with n frequencies as periodic functions of functions with $n-1$ frequencies, which constitute a function space. Then we define Poincare′s return map of a given semiflow on the space whose fixed point corresponds to an invariant torus of the semiflow.

1. **Introduction.** Poincare's map has been a basic and powerful concept for the study of differential equations. For the non-autonomous equations with period T, a fixed point of time-T map in the phase space corresponds to a periodic solution with period T. It is natural to ask whether one can generalize this idea to quasi-periodic equations. One way to do it is to consider the equation as defined on $\mathbf{T}^n \times X$, where $\mathbf{T} \equiv \mathbf{R}/\mathbf{Z}$ and X is the phase space of the equation, and take $\mathbf{T}^{n-1} \times \{0\} \times X$ as a Poincare's section. Starting with any function on $\mathbf{T}^{n-1}$, we can define the first return map on the function space by integrating the equation. A fixed point of this map corresponds to a quasi-periodic solution to the equation. This idea is to regard a quasiperiodic function with n base frequencies as a periodic function of the quasiperiodic functions with $n-1$ base frequencies. In the periodic Poincare's map, fixed point may be obtained through contraction or monotone iteration. The latter has given fruitful results in the study of order-preserving compact flows, arising from reaction-diffusion equations for example, see [4]. Applying this idea to the quasiperiodic problem, a stark difference appears already in the first-order ODE because of lack of compactness in general. Hence we expect to obtain certain discontinuous quasi-periodic solutions.

In the Section 2 of this paper, we present basic setups of the method and a contractional condition under which the fixed point exists. In Section 3 we apply the method to first-order ODEs. Starting from an upper or lower equilibrium, we obtain a monotone sequence of continuous functions that converges to a fixed point of the map. The invariant torus thus constructed is semi-continuous and belongs to the weak quasiperiodic solutions defined and studied in [8]. Hence by their theorems it contains almost-automorphic solutions on its continuous points. Conversely the intersection of any weak quasiperiodic solution with the Poincare section is a semi-continuous fixed point of the Poincare's map. By analyzing the dynamics on the

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